High-Rate Quantization for the Neyman-Pearson Detection of Hidden Markov Processes

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ITW 2010
Context

- a physical phenomenon with space correlation
- some sensors
- a fusion center
- wireless channels
  → quantization

Goal: detection from quantized observations
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- some sensors
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Goal: detection from quantized observations

Questions:
- performance of the Neyman-Pearson test?
- best quantization?
Neyman-Pearson test on quantized observations

$n$ sensors
quantization on $\log_2(N)$ bits

Evaluation of the performance:

$$P_e \approx e^{-n\left(K - \frac{D}{N^2}\right)}$$

for large $n, N$ and $n \gg N$. 
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Our aims:

- Evaluate the loss $D$ due to quantization
- Find the best quantization rule
Outline

1. Detection from Unquantized Observations
2. Detection from Quantized Observations
3. Detection in the High-Rate Regime
Neyman-Pearson Hypothesis Testing

- $Y_{1:n} = (Y_1 \ldots Y_n)$: a **stationary** real-valued Lebesgue-dominated process with **mixing** properties

- Binary test
  
  $H_0 : Y_{1:n} \sim p_0$
  
  $H_1 : Y_{1:n} \sim p_1$
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Neyman-Pearson strategy:

- Set \( P_{H_0} (\text{decide } H_1) = \alpha \) **false alarm**
- Minimize \( P_{H_1} (\text{decide } H_0) \rightarrow \beta_n(\alpha) \) **miss**

Likelihood Ratio Test:

\[
L_n = \frac{1}{n} \log \frac{p_0}{p_1}(Y_{1:n}) \begin{cases} H_0 \\ H_1 \end{cases} \geq \lambda_n
\]
Error Exponent

Our aim is to measure the detection performance.

- $\beta_n(\alpha)$ is a good performance measure . . .
- . . . but is not tractable

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Lemma (Stein – Chen)

If $\exists K > 0$ such that $L_n \xrightarrow{P} K$ under $H_0$ then

$$\forall \alpha \in (0, 1) \quad \lim_{n \to +\infty} \frac{1}{n} \log \beta_n(\alpha) = -K$$

$K$ is the error exponent of the test: $\beta_n(\alpha) \approx \exp(-nK)$
Error Exponent for Unquantized Observations ($n \to \infty$)

Under a certain mixing condition on $p_1$
(e.g. valid for a wide class of hidden Markov models)

*Shannon-McMillan-Breiman*–like result

The LLR $L_n$ converges under $H_0$ to

$$K = E_0 \left[ \log \frac{p_0}{p_1} (Y_0 | Y_{-\infty:-1}) \right]$$

$K$ is the error exponent of the NP test.
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Neyman-Pearson Test on Quantized Observations

- Quantized observation: \( Z_{N,k} = Q_N(Y_k) \)
- The test becomes:
  \[
  H_0 : Z_{N,1:n} \sim p_{0,N} \\
  H_1 : Z_{N,1:n} \sim p_{1,N}
  \]

Error exponent

\[
K_N = E_0 \left[ \log \frac{p_{0,N}}{p_{1,N}} (Z_{N,0}|Z_{N,-\infty:-1}) \right]
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K_N = E_0 \left[ \log \frac{p_{0,N}}{p_{1,N}} \left( Z_{N,0} | Z_{N,-\infty:-1} \right) \right]
\]

Our aim is to study the error exponent loss \( K - K_N \)

\( \rightarrow \) [Gupta & Hero – 2003] for i.i.d. observations.
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High-Rate Quantization ($N \rightarrow \infty$)

Asymptotic regime: \( n, N \rightarrow \infty \text{ but } n \gg N \)

cf. [Bennett48], [Gray98]
High-Rate Quantization \((N \to \infty)\)

Asymptotic regime: \(n, N \to \infty\) but \(n \gg N\)

cf. [Bennett48], [Gray98]

Model point density \(\zeta\)

\(\approx\) asymptotic number of cells in the neighborhood of \(y\)

In the high-rate regime:

\[
\frac{\text{number of quantization points in } A}{N} \Rightarrow \int_A \zeta(y) \, dy
\]
Main Result

Theorem (Asymptotic Error Exponent Loss)

\[ N^2(K - K_N) \xrightarrow{N \to \infty} D\xi = \frac{1}{24} \int p_0(y) F(y) \zeta(y)^2 \, dy \]

where \( F(y) = E_0 \left[ \left( \frac{\partial}{\partial y_0} \log \frac{p_0}{p_1} (Y_{-\infty:\infty}) \right)^2 \right] \mid Y_0 = y \)

Under some mixing conditions:

\[ \eta_m^{-1} \leq \frac{p_i(Y_0|Y_{-m':-1})}{p_i(Y_0|Y_{-m:-1})} \leq \eta_m, \quad \eta_m^{-1} \leq \frac{p_{i,N}(Z_{N,0}|Z_{N,-m':-1})}{p_{i,N}(Z_{N,0}|Z_{N,-m:-1})} \leq \eta_m \]

\[ \left| \frac{\partial}{\partial y_0} \log p_i(Y_{0:k}|Y_{-\ell:-1}) - \frac{\partial}{\partial y_0} \log p_i(Y_{0:k}|Y_{-\ell':-1}) \right| \leq \varphi_\ell \]

\[ \left| \frac{\partial}{\partial y_0} \log p_i(Y_k|Y_{-\ell:k-1}) \right| \leq \psi_k \]

for \( \log \eta_m = O(m^{-6-\epsilon}) \) and some summable \( \varphi_k \) and \( \psi_k \).
Key Ideas of the Proof

- Inversion of two limits:

\[
\lim_{N \to \infty} N^2(K - K_N) = \lim_{N \to \infty} \lim_{m \to \infty} N^2 E_0 \left[ \log \frac{p_0}{p_1} (Y_0 | Y_{-m:1}) - \log \frac{p_{0,N}}{p_{1,N}} (Z_N,0 | Z_N,-m:1) \right]
\]

- Taylor-Lagrange expansion of densities: \( \frac{p_{0,N}}{p_{1,N}} \approx \frac{p_0}{p_1} \) as \( N \to \infty \)

Main issue:
Find relevant estimates of the remainders in \( m, N \).
\( \to \) Mixing conditions are needed.
Detection of a Gaussian AR-1 Process in Noise

State:

\[ X_k = aX_{k-1} + \sqrt{1 - a^2} U_k \]

Observation:

\[ H_0 : Y_k = W_k \]
\[ H_1 : Y_k = X_k + W_k \]

- \( a \in (0, 1) \): correlation coefficient
- \( U_k \) i.i.d. \( \mathcal{N}(0, 1) \): innovation
- \( W_k \) i.i.d. \( \mathcal{N}(0, \sigma^2) \): obs. noise
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Probability and model point densities (\( \sigma = 1 \))
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Probability and model point densities \((\sigma = 1)\)

\[ D_\xi = f(a) \text{ for different quantization strategies } (\sigma = 1) \]
Conclusion

- Neyman-Pearson test on quantized observations → \( n \) sensors, quantization on \( \log_2(N) \) bits

- Evaluation of the performance:

\[
\beta_n(\alpha) \approx e^{-n\left(K - \frac{D}{N^2}\right)}
\]

for large \( n, N \) and \( n \gg N \).

- **Optimal** quantization rule

→ Valid for a wide class of stationary mixing processes
Ongoing Work: Vector Quantization (to be sub. to ISIT2010)

Vector-valued process: \( Y_k \in Y \subset \mathbb{R}^d \)

Theorem (Asymptotic Error Exponent Loss)
Under some mixing conditions:

\[
N^{2/d}(K - K_N) \xrightarrow{N \to \infty} D_e = \frac{1}{2} \int \frac{p_0(y) F(y)}{\zeta(y)^{2/d}} \, dy ,
\]

where \( F(y) = \mathbb{E}_0 \left[ \nabla_{y_0} \log \frac{p_0}{p_1} (Y_{-\infty:\infty})^T M(Y_0) \nabla_{y_0} \log \frac{p_0}{p_1} (Y_{-\infty:\infty}) \mid Y_0 = y \right] \)
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$M$ is the model covariation profile:
- a matrix-valued function...
- ...which provides information about the shape of the cells
Thank you for your attention.